1. Suppose we want to write an efficient function SHUFFLE(n) that returns a permutation of the set $\{1, 2, \ldots, n\}$ chosen uniformly at random.

   (a) Prove that the following algorithm is ***not*** correct. *[Hint: Consider the case $n = 3$.]*

```
SHUFFLE(n):
  for i ← 1 to n
      π[i] ← i
  for i ← 1 to n
      swap π[i] ↔ π[RANDOM(n)]
  return π[1..n]
```

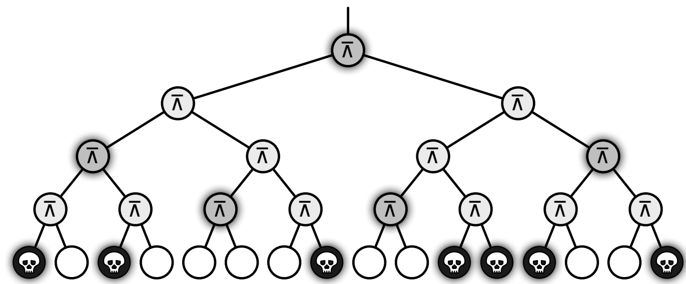
   (b) Consider the following implementation of SHUFFLE.

```
SHUFFLE(n):
  for i ← 1 to n
      π[i] ← NULL
  for i ← 1 to n
      j ← RANDOM(n)
      while (π[j] != NULL)
          j ← RANDOM(n)
      π[j] ← i
  return π[1..n]
```

   Prove that this algorithm is correct. What is its expected running time?

   (c) Describe and analyze an implementation of SHUFFLE that runs in $O(n)$ time. (An algorithm that runs in $O(n)$ *expected* time is fine, but $O(n)$ *worst-case* time is possible.)

2. Death knocks on your door one cold blustery morning and challenges you to a game. Death knows you are an algorithms student, so instead of the traditional game of chess, Death presents you with a complete binary tree with 4^n leaves, each colored either black or white. There is a token at the root of the tree. To play the game, you and Death will take turns moving the token from its current node to one of its children. The game will end after $2n$ moves, when the token lands on a leaf. If the final leaf is black, you die; if it's white, you will live forever. You move first, so Death gets the last turn.



   You can decide whether it's worth playing or not as follows. Imagine that the tree is a Boolean circuit whose inputs are specified at the leaves: white and black represent TRUE and FALSE inputs, respectively. Each internal node in the tree is a NAND gate that gets its input from its children and passes its output to its parent. (Recall that a NAND gate outputs FALSE if and only if both its inputs are TRUE.) If the output at the top of the tree is TRUE, then you can win and live forever! If the output at the top of the tree is FALSE, you should challenge Death to a game of Twister instead. Or maybe Battleship.

(a) Describe and analyze a deterministic algorithm to determine whether or not you can win. *[Hint: This is easy!]*

(b) Unfortunately, Death won't give you enough time to look at every node in the tree. Describe a *randomized* algorithm that determines whether you can win in $O(3^n)$ expected time. *[Hint: Consider the case $n = 1$.]*

*(c) **[Extra credit]** Describe and analyze a randomized algorithm that determines whether you can win in $O(c^n)$ expected time, for some constant $c < 3$. *[Hint: You may not need to change your algorithm from part (b) at all!]*

3. A *meldable priority queue* stores a set of keys from some totally-ordered universe (such as the integers) and supports the following operations:

- MAKEQUEUE: Return a new priority queue containing the empty set.
- FINDMIN(Q): Return the smallest element of Q (if any).
- DELETEMIN(Q): Remove the smallest element in Q (if any).
- INSERT(Q, x): Insert element x into Q, if it is not already there.
- DECREASEKEY(Q, x, y): Replace an element $x \in Q$ with a smaller key y. (If $y > x$, the operation fails.) The input is a pointer directly to the node in Q containing x.
- DELETE(Q, x): Delete the element $x \in Q$. The input is a pointer directly to the node in Q containing x.
- MELD(Q_1, Q_2): Return a new priority queue containing all the elements of Q_1 and Q_2; this operation destroys Q_1 and Q_2.

A simple way to implement such a data structure is to use a heap-ordered binary tree, where each node stores a key, along with pointers to its parent and two children. MELD can be implemented using the following randomized algorithm:

```
MELD(Q1, Q2):
  if Q1 is empty return Q2
  if Q2 is empty return Q1
  if key(Q1) > key(Q2)
      swap Q1 ↔ Q2
  with probability 1/2
      left(Q1) ← MELD(left(Q1), Q2)
  else
      right(Q1) ← MELD(right(Q1), Q2)
  return Q1
```

(a) Prove that for *any* heap-ordered binary trees Q_1 and Q_2 (not just those constructed by the operations listed above), the expected running time of MELD(Q_1, Q_2) is $O(\log n)$, where $n = |Q_1| + |Q_2|$. *[Hint: How long is a random root-to-leaf path in an n-node binary tree if each left/right choice is made with equal probability?]*

(b) Show that each of the other meldable priority queue operations can be implemented with at most one call to MELD and $O(1)$ additional time. (This implies that every operation takes $O(\log n)$ expected time.)