> This exam lasts 180 minutes.
> **Write your answers in the separate answer booklet.**
> Please return this question sheet with your answers.

1. Choreographer Michael Flatley has hired a new dance company to perform his latest Irish step-dancing extravaganza. At their first practice session, the new dancers line up in a row on stage and practice a movement called the *Flatley Flip*: Whenever Mr. Flatley calls out any positive integer k, the k rightmost dancers rotate 180 degrees as a group, so that their order in the line is reversed.

   Each dancer wears a shirt with a positive integer printed on the front and back; different dancers have different numbers. Mr. Flatley wants to rearrange the dancers, using only a sequence of Flatley Flips, so that these numbers are sorted from left to right in increasing order.
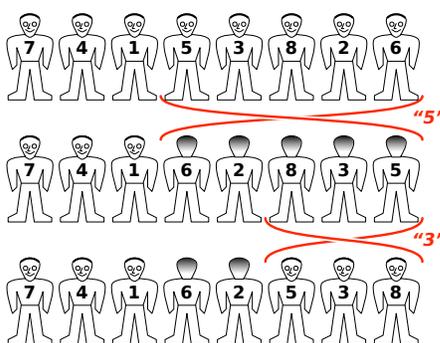
   Two Flatley flips.

   (a) Describe an algorithm to sort an arbitrary row of n numbered dancers, using $O(n)$ Flatley flips. (After sorting, the dancers may face forward, backward, or some of each.) *Exactly* how many flips does your algorithm perform in the worst case?[1]

   (b) Describe an algorithm that sorts an arbitrary row of n numbered dancers *and ensures that all dancers are facing forward*, using $O(n)$ Flatley flips. *Exactly* how many flips does your algorithm perform in the worst case?[2]

2. You're in charge of choreographing a musical for your local community theater, and it's time to figure out the final pose of the big show-stopping number at the end. ("Streetcar!") You've decided that each of the n cast members in the show will be positioned in a big line when the song finishes, all with their arms extended and showing off their best spirit fingers.

   The director has declared that during the final flourish, each cast member must either point both their arms up or point both their arms down; it's your job to figure out who points up and who points down. Moreover, in a fit of unchecked power, the director has also given you a list of arrangements that will upset his delicate artistic temperament. Each forbidden arrangement is a subset of cast members paired with arm positions; for example: "Marge may not point her arms up while Ned and Apu point their arms down."

   Prove that finding an acceptable arrangement of arm positions is NP-hard. *[Hint: Describe a reduction from 3SAT.]*

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[1]This is really a question about networking.
[2]This is really a question about mutating DNA.

3. **Dance Dance Revolution** is a dance video game, first introduced in Japan by Konami in 1998. Players stand on a platform marked with four arrows, pointing forward, back, left, and right, arranged in a cross pattern. During play, the game plays a song and scrolls a sequence of n arrows (←, ↑, ↓, or →) from the bottom to the top of the screen. At the precise moment each arrow reaches the top of the screen, the player must step on the corresponding arrow on the dance platform. (The arrows are timed so that you'll step with the beat of the song.)

   You are playing a variant of this game called "Vogue Vogue Revolution", where the goal is to play perfectly but move as little as possible. When an arrow reaches the top of the screen, if one of your feet is already on the correct arrow, you are awarded one style point for maintaining your current pose. If neither foot is on the right arrow, you must move one (and *only* one) of your feet from its current location to the correct arrow on the platform. If you ever step on the wrong arrow, or fail to step on the correct arrow, or move more than one foot at a time, all your style points are taken away and the games ends.

   How should you move your feet to maximize your total number of style points? For purposes of this problem, assume you always start with you left foot on ← and you right foot on →, and that you've memorized the entire sequence of arrows. For example, if the sequence is ↑↑↓↓←→←→, you can earn 5 style points by moving you feet as shown below:

   Begin!   Style point!   Style point!   Style point!   Style point!   Style point!
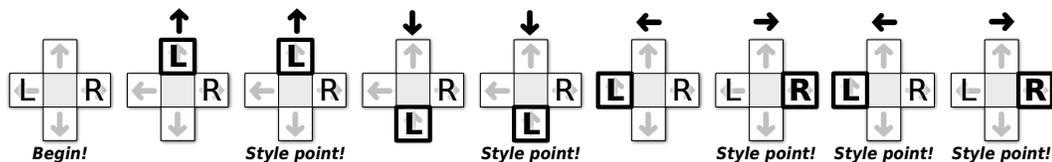
   (a) **Prove** that for *any* sequence of n arrows, it is possible to earn at least $n/4 - 1$ style points.

   (b) Describe an efficient algorithm to find the maximum number of style points you can earn during a given VVR routine.[3] The input to your algorithm is an array $Arrow[1..n]$ containing the sequence of arrows. *[Hint: Build a graph!]*

4. It's almost time to show off your flippin' sweet dancing skills! Tomorrow is the big dance contest you've been training for your entire life, except for that summer you spent with your uncle in Alaska hunting wolverines. You've obtained an advance copy of the the list of n songs that the judges will play during the contest, in chronological order.

   You know all the songs, all the judges, and your own dancing ability extremely well. For each integer k, you know that if you dance to the kth song on the schedule, you will be awarded exactly $Score[k]$ points, but then you will be physically unable to dance for the next $Wait[k]$ songs (that is, you cannot dance to songs $k + 1$ through $k + Wait[k]$). The dancer with the highest total score at the end of the night wins the contest, so you want your total score to be as high as possible.

   Describe and analyze an efficient algorithm to compute the maximum total score you can achieve. The input to your sweet algorithm is the pair of arrays $Score[1..n]$ and $Wait[1..n]$.[4]

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   [3]This is really a question about paging.
   [4]This is really a question about processor scheduling.

5. You're organizing the First Annual UIUC Computer Science 72-Hour Dance Exchange, to be held all day Friday, Saturday, and Sunday. Several 30-minute sets of music will be played during the event, and a large number of DJs have applied to perform. You need to hire DJs according to the following constraints.

   - Exactly k sets of music must be played each day, and thus $3k$ sets altogether.
   - Each set must be played by a single DJ in a consistent music genre (ambient, bubblegum, dubstep, horrorcore, hyphy, trip-hop, Nitzhonot, Kwaito, J-pop, Nashville country, ...).
   - Each genre must be played at most once per day.
   - Each candidate DJ has given you a list of genres they are willing to play.
   - Each DJ can play at most three sets during the entire event.

   Suppose there are n candidate DJs and g different musical genres available. Describe and analyze an efficient algorithm that either assigns a DJ and a genre to each of the $3k$ sets, or correctly reports that no such assignment is possible.

6. You've been put in charge of putting together a team for the "Dancing with the Computer Scientists" international competition. Good teams in this competition must be capable of performing a wide variety of dance styles. You are auditioning a set of n dancing computer scientists, each of whom specializes in a particular style of dance.

   Describe an algorithm to determine in $O(n)$ time if *more than half* of the n dancers specialize exactly in the same dance style. The input to your algorithm is an array of n positive integers, where each integer identifies a style: $1 =$ ballroom, $2 =$ latin, $3 =$ swing, $4 =$ b-boy, $42 =$ contact improv, $101 =$ peanut butter jelly time, and so on. *[Hint: Remember the Select algorithm!]*

7. The party you are attending is going great, but now it's time to line up for ***The Algorithm March (アルゴリズムこうしん)***! This dance was originally developed by the Japanese comedy duo Itsumo Kokokara (いつもここから) for the children's television show PythagoraSwitch (ピタゴラスイッチ). The Algorithm March is performed by a line of people; each person in line starts a specific sequence of movements one measure later than the person directly in front of them. Thus, the march is the dance equivalent of a musical round or canon, like "Row Row Row Your Boat".

   Proper etiquette dictates that each marcher must know the person directly in front of them in line, lest a minor mistake during lead to horrible embarrassment between strangers. Suppose you are given a complete list of which people at your party know each other. ***Prove*** that it is NP-hard to determine the largest number of party-goers that can participate in the Algorithm March.[5]

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[5]This is really a question about ninjas.

You may assume the following problems are NP-hard:

CIRCUITSAT: Given a boolean circuit, are there any input values that make the circuit output True?

PLANARCIRCUITSAT: Given a boolean circuit drawn in the plane so that no two wires cross, are there any input values that make the circuit output True?

3SAT: Given a boolean formula in conjunctive normal form, with exactly three literals per clause, does the formula have a satisfying assignment?

MAXINDEPENDENTSET: Given an undirected graph G, what is the size of the largest subset of vertices in G that have no edges among them?

MAXCLIQUE: Given an undirected graph G, what is the size of the largest complete subgraph of G?

MINVERTEXCOVER: Given an undirected graph G, what is the size of the smallest subset of vertices that touch every edge in G?

MINDOMINATINGSET: Given an undirected graph G, what is the size of the smallest subset S of vertices such that every vertex in G is either in S or adjacent to a vertex in S?

MINSETCOVER: Given a collection of subsets $S_1, S_2, \ldots, S_m$ of a set S, what is the size of the smallest subcollection whose union is S?

MINHITTINGSET: Given a collection of subsets $S_1, S_2, \ldots, S_m$ of a set S, what is the size of the smallest subset of S that intersects every subset S_i?

3COLOR: Given an undirected graph G, can its vertices be colored with three colors, so that every edge touches vertices with two different colors?

CHROMATICNUMBER: Given an undirected graph G, what is the minimum number of colors needed to color its vertices, so that every edge touches vertices with two different colors?

MAXCUT: Given a graph G, what is the size (number of edges) of the largest bipartite subgraph of G?

HAMILTONIANCYCLE: Given a graph G, is there a cycle in G that visits every vertex exactly once?

HAMILTONIANPATH: Given a graph G, is there a path in G that visits every vertex exactly once?

TRAVELINGSALESMAN: Given a graph G with weighted edges, what is the minimum total weight of any Hamiltonian path/cycle in G?

SUBSETSUM: Given a set X of positive integers and an integer k, does X have a subset whose elements sum to k?

PARTITION: Given a set X of positive integers, can X be partitioned into two subsets with the same sum?

3PARTITION: Given a set X of n positive integers, can X be partitioned into $n/3$ three-element subsets, all with the same sum?

MINESWEEPER: Given a Minesweeper configuration and a particular square x, is it safe to click on x?

TETRIS: Given a sequence of N Tetris pieces and a partially filled $n \times k$ board, is it possible to play every piece in the sequence without overflowing the board?

SUDOKU: Given an $n \times n$ Sudoku puzzle, does it have a solution?

KENKEN: Given an $n \times n$ Ken-Ken puzzle, does it have a solution?