CS 373U: Combinatorial Algorithms, Spring 2004
Homework 0

Due January 28, 2004 at noon

Name:	
Net ID:	Alias:

☐ I understand the Homework Instructions and FAQ.

- Neatly print your full name, your NetID, and an alias of your choice in the boxes above. Grades will be listed on the course web site by alias; for privacy reasons, your alias should not resemble your name or NetID. By providing an alias, you agree to let us list your grades; if you do not provide an alias, your grades will not be listed. ***Never*** *give us your Social Security number!*

- Before you do anything else, read the Homework Instructions and FAQ on the course web page, and then check the box above. This web page gives instructions on how to write and submit homeworks—staple your solutions together in order, start each numbered problem on a new sheet of paper, write your name and NetID one every page, don't turn in source code, analyze and prove everything, use good English and good logic, and so on. See especially the policies regarding the magic phrases "I don't know" and "and so on". If you have *any* questions, post them to the course newsgroup or ask in lecture.

- This homework tests your familiarity with prerequisite material—basic data structures, big-Oh notation, recurrences, discrete probability, and most importantly, induction—to help you identify gaps in your knowledge. **You are responsible for filling those gaps on your own.** Chapters 1–10 of CLRS should be sufficient review, but you may also want consult your discrete mathematics and data structures textbooks.

- Every homework will have five required problems and one extra-credit problem. Each numbered problem is worth 10 points.

#	1	2	3	4	5	6*	Total
Score							
Grader							

1. Sort the functions in each box from asymptotically smallest to asymptotically largest, indicating ties if there are any. You do not need to turn in proofs (in fact, please *don't* turn in proofs), but you should do them anyway, just for practice. Don't merge the lists together.

   To simplify your answers, write $f(n) \ll g(n)$ to mean $f(n) = o(g(n))$, and write $f(n) \equiv g(n)$ to mean $f(n) = \Theta(g(n))$. For example, the functions $n^2, n, \binom{n}{2}, n^3$ could be sorted either as $n \ll n^2 \equiv \binom{n}{2} \ll n^3$ or as $n \ll \binom{n}{2} \equiv n^2 \ll n^3$.

   (a)

$2^{\sqrt{\lg n}}$	$2^{\lg \sqrt{n}}$	$\sqrt{2^{\lg n}}$	$\sqrt{2}^{\lg n}$	$\lg 2^{\sqrt{n}}$	$\lg \sqrt{2}^{n}$	$\lg \sqrt{2^n}$	$\sqrt{\lg 2^n}$
$\lg n^{\sqrt{2}}$	$\lg \sqrt{n}^2$	$\lg \sqrt{n^2}$	$\sqrt{\lg n^2}$	$\lg^2 \sqrt{n}$	$\lg^{\sqrt{2}} n$	$\sqrt{\lg^2 n}$	$\sqrt{\lg n}^2$

   ⋆(b)

$\lg(\sqrt{n}!)$	$\lg(\sqrt{n!})$	$\sqrt{\lg(n!)}$	$(\lg \sqrt{n})!$	$(\sqrt{\lg n})!$	$\sqrt{(\lg n)!}$

   [Hint: Use Stirling's approximation for factorials: $n! \approx n^{n+1/2}/e^n$]

2. Solve the following recurrences. State tight asymptotic bounds for each function in the form $\Theta(f(n))$ for some recognizable function $f(n)$. Proofs are *not* required; just give us the list of answers. You do not need to turn in proofs (in fact, please *don't* turn in proofs), but you should do them anyway, just for practice. Assume reasonable but nontrivial base cases. If your solution requires specific base cases, state them! Extra credit will be awarded for more exact solutions.

   (a) $A(n) = 9A(n/3) + n^2$

   (b) $B(n) = 2B(n/2) + n/\lg n$

   (c) $C(n) = \dfrac{2C(n-1)}{C(n-2)}$ *[Hint: This is easy!]*

   (d) $D(n) = D(n-1) + 1/n$

   (e) $E(n) = E(n/2) + D(n)$

   (f) $F(n) = 2F\big(\lfloor (n+3)/4 \rfloor - \sqrt{5n \lg n} + 6\big) + 7\sqrt{n+8} - \lg^9 \lg \lg n + 10^{\lg^* n} - 11/n^{12}$

   (g) $G(n) = 3G(n-1) - 3G(n-2) + G(n-3)$

   ⋆(h) $H(n) = 4H(n/2) - 4H(n/4) + 1$ *[Hint: Careful!]*

   (i) $I(n) = I(n/3) + I(n/4) + I(n/6) + I(n/8) + I(n/12) + I(n/24) + n$

   ★(j) $J(n) = \sqrt{n} \cdot J(2\sqrt{n}) + n$
   [Hint: First solve the secondary recurrence $j(n) = 1 + j(2\sqrt{n})$.]

3. Scientists have recently discovered a planet, tentatively named "Ygdrasil", which is inhabited by a bizarre species called "nertices" (singular "nertex"). All nertices trace their ancestry back to a particular nertex named Rudy. Rudy is still quite alive, as is every one of his many descendants. Nertices reproduce asexually, like bees; each nertex has exactly one parent (except Rudy). There are three different types of nertices—red, green, and blue. The color of each nertex is correlated exactly with the number and color of its children, as follows:

   - Each red nertex has two children, exactly one of which is green.
   - Each green nertex has exactly one child, which is not green.
   - Blue nertices have no children.

   In each of the following problems, let R, G, and B respectively denote the number of red, green, and blue nertices on Ygdrasil.

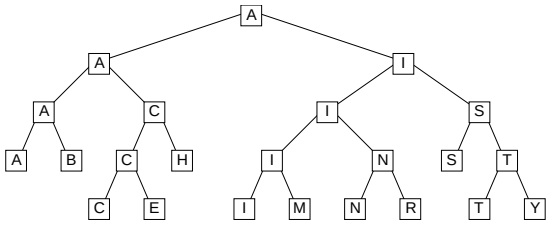
   (a) Prove that $B = R + 1$.

   (b) Prove that either $G = R$ or $G = B$.

   (c) Prove that $G = B$ if and only if Rudy is green.

4. Algorithms and data structures were developed millions of years ago by the Martians, but not quite in the same way as the recent development here on Earth. Intelligent life evolved independently on Mars' two moons, Phobos and Deimos.[1] When the two races finally met on the surface of Mars, after thousands of years of separate philosophical, cultural, religious, and scientific development, their disagreements over the proper structure of binary search trees led to a bloody (or more accurately, ichorous) war, ultimately leading to the destruction of all Martian life.

   A *Phobian* binary search tree is a full binary tree that stores a set X of search keys. The root of the tree stores the *smallest* element in X. If X has more than one element, then the left subtree stores all the elements less than some pivot value p, and the right subtree stores everything else. Both subtrees are *nonempty* Phobian binary search trees. The actual pivot value p is *never* stored in the tree.



A Phobian binary search tree for the set {M, A, R, T, I, N, B, Y, S, E, C, H}.

   (a) Describe and analyze an algorithm FIND(x, T) that returns TRUE if x is stored in the Phobian binary search tree T, and FALSE otherwise.

   (b) A *Deimoid* binary search tree is almost exactly the same as its Phobian counterpart, except that the *largest* element is stored at the root, and both subtrees are Deimoid binary search trees. Describe and analyze an algorithm to transform an n-node Phobian binary search tree into a Deimoid binary search tree in $O(n)$ time, using as little additional space as possible.

[1] Greek for "fear" and "panic", respectively. Doesn't that make you feel better?

5. Penn and Teller agree to play the following game. Penn shuffles a standard deck[2] of playing cards so that every permutation is equally likely. Then Teller draws cards from the deck, one at a time without replacement, until he draws the three of clubs (3♣), at which point the remaining undrawn cards instantly burst into flames.

   The first time Teller draws a card from the deck, he gives it to Penn. From then on, until the game ends, whenever Teller draws a card whose value is smaller than the last card he gave to Penn, he gives the new card to Penn.[3] To make the rules unambiguous, they agree beforehand that $A = 1$, $J = 11$, $Q = 12$, and $K = 13$.

   (a) What is the expected number of cards that Teller draws?

   (b) What is the expected *maximum* value among the cards Teller gives to Penn?

   (c) What is the expected *minimum* value among the cards Teller gives to Penn?

   (d) What is the expected number of cards that Teller gives to Penn?

   Full credit will be given only for *exact* answers (with correct proofs, of course).

*6. **[Extra credit]**[4]

   Lazy binary is a variant of standard binary notation for representing natural numbers where we allow each "bit" to take on one of three values: 0, 1, or 2. Lazy binary notation is defined inductively as follows.

   - The lazy binary representation of zero is 0.
   - Given the lazy binary representation of any non-negative integer n, we can construct the lazy binary representation of $n + 1$ as follows:
     (a) increment the rightmost digit;
     (b) if any digit is equal to 2, replace the rightmost 2 with 0 and increment the digit immediately to its left.

   Here are the first several natural numbers in lazy binary notation:

   0, 1, 10, 11, 20, 101, 110, 111, 120, 201, 210, 1011, 1020, 1101, 1110, 1111, 1120, 1201, 1210, 2011, 2020, 2101, 2110, 10111, 10120, 10201, 10210, 11011, 11020, 11101, 11110, 11111, 11120, 11201, 11210, 12011, 12020, 12101, 12110, 20111, 20120, 20201, 20210, 21011, 21020, 21101, 21110, 101111, 101120, 101201, 101210, 102011, 102020, 102101, 102110, ...

   (a) Prove that in any lazy binary number, between any two 2s there is at least one 0, and between two 0s there is at least one 2.

   (b) Prove that for any natural number N, the sum of the digits of the lazy binary representation of N is exactly $\lfloor \lg(N + 1) \rfloor$.

[2] In a standard deck of 52 cards, each card has a *suit* in the set $\{\spadesuit, \heartsuit, \clubsuit, \diamondsuit\}$ and a *value* in the set $\{A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K\}$, and every possible suit-value pair appears in the deck exactly once. Actually, to make the game more interesting, Penn and Teller normally use razor-sharp ninja throwing cards.

[3] Specifically, he hurls them from the opposite side of the stage directly into the back of Penn's right hand.

[4] The "I don't know" rule does not apply to extra credit problems. There is no such thing as "partial extra credit".