CS 473: Undergraduate Algorithms, Fall 2013

Headbanging 0: Induction!

August 28 and 29

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1. Prove that any non-negative integer can be represented as the sum of distinct powers of 2. ("Write it in binary" is not a proof; it's just a restatement of what you have to prove.)

2. Prove that every integer (positive, negative, or zero) can be written in the form $\sum_i \pm 3^i$, where the exponents i are distinct non-negative integers. For example:

$$42 = 3^4 - 3^3 - 3^2 - 3^1 \qquad 25 = 3^3 - 3^1 + 3^0 \qquad 17 = 3^3 - 3^2 - 3^0$$

3. Recall that a ***full binary tree*** is either an isolated *leaf*, or an *internal node* with a left subtree and a right subtree, each of which is a full binary tree. Equivalently, a binary tree is ***full*** if every internal node has exactly two children. Give at least *three different* proofs of the following fact: *In every full binary tree, the number of leaves is exactly one more than the number of internal nodes.*

Take-home points:

- Induction is recursion. Recursion is induction.
- All induction is strong/structural induction. There is absolutely no point in using a *weak* induction hypothesis. None. Ever.
- To prove that all snarks are boojums, start with an *arbitrary* snark and remove some tentacles. Do not start with a smaller snark and try to add tentacles. Snarks don't like that.
- Every induction proof requires an exhaustive case analysis. Write down the cases. Make sure they're exhaustive.
- Do the most general cases first. Whatever is left over are the base cases.
- The empty set is the best base case.

Khelm is Warsaw. Warsaw is Khelm. Khelm is Warsaw. Zay gezunt!
Warsaw is Khelm. Khelm is Warsaw. Warsaw is Khelm. For gezunt!

— Golem (feat. Amanda Palmer), "Warsaw is Khelm", *Fresh Off Boat* (2006)