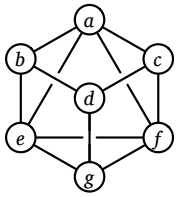
This exam lasts 90 minutes.
Write your answers in the separate answer booklet.
Please return this question sheet with your answers.

1. (a) Suppose $A[1..n]$ is an array of n distinct integers, sorted so that $A[1] < A[2] < \cdots < A[n]$. Each integer $A[i]$ could be positive, negative, or zero. Describe and analyze an efficient algorithm that either computes an index i such that $A[i] = i$ or correctly reports that no such index exists.

   (b) Now suppose $A[1..n]$ is a sorted array of n distinct ***positive*** integers. Describe and analyze an even faster algorithm that either computes an index i such that $A[i] = i$ or correctly reports that no such index exists.

2. A *double-Hamiltonian circuit* a closed walk in a graph that visits every vertex exactly *twice*. ***Prove*** that it is NP-hard to determine whether a given graph contains a double-Hamiltonian circuit.



This graph contains the double-Hamiltonian circuit $a \to b \to d \to g \to e \to b \to d \to c \to f \to a \to c \to f \to g \to e \to a$.

3. A palindrome is any string that is exactly the same as its reversal, like `I`, or `DEED`, or `HANNAH`, or `AMANAPLANACATACANALPANAMA`. Describe and analyze an algorithm to find the length of the longest subsequence of a given string that is also a palindrome.

   For example, the longest palindrome subsequence of `MAHDYNAMICPROGRAMZLETMESHOWYOUTHEM` is `MHYMRORMYHM`, so given that string as input, your algorithm should return the integer 11.

4. Suppose you are given a magic black box that can determine **in polynomial time**, given an arbitrary graph G, the number of vertices in the largest complete subgraph of G. Describe and analyze a **polynomial-time** algorithm that computes, given an arbitrary graph G, a complete subgraph of G of maximum size, using this magic black box as a subroutine.

5. Suppose we are given a $4 \times n$ grid, where each grid cell has an integer value. Suppose we want to mark a subset of the grid cells, so that the total value of the marked cells is as large as possible. However, we are forbidden to mark any pair of grid cells that are immediate horizontal or vertical neighbors. (Marking diagonal neighbors is fine.) Describe and analyze an algorithm that computes the largest possible sum of marked cells, subject to this non-adjacency condition.

   For example, given the grid on the left below, your algorithm should return the integer 36, which is the sum of the circled numbers on the right.

4	−5	1	6
2	6	−1	8
5	4	3	3
1	−1	7	4
−3	4	5	−2

$\Longrightarrow$

④	−5	1	⑥
2	⑥	−1	8
⑤	4	3	③
1	−1	⑦	4
−3	⑤	4	−2

You may assume the following problems are NP-hard:

CircuitSat: Given a boolean circuit, are there any input values that make the circuit output True?

PlanarCircuitSat: Given a boolean circuit drawn in the plane so that no two wires cross, are there any input values that make the circuit output True?

3Sat: Given a boolean formula in conjunctive normal form, with exactly three literals per clause, does the formula have a satisfying assignment?

Max2Sat: Given a boolean formula in conjunctive normal form, with exactly two literals per clause, what is the largest number of clauses that can be satisfied by an assignment?

MaxIndependentSet: Given an undirected graph G, what is the size of the largest subset of vertices in G that have no edges among them?

MaxClique: Given an undirected graph G, what is the size of the largest complete subgraph of G?

MinVertexCover: Given an undirected graph G, what is the size of the smallest subset of vertices that touch every edge in G?

MinDominatingSet: Given an undirected graph G, what is the size of the smallest subset S of vertices such that every vertex in G is either in S or adjacent to a vertex in S?

MinSetCover: Given a collection of subsets $S_1, S_2, \ldots, S_m$ of a set S, what is the size of the smallest subcollection whose union is S?

MinHittingSet: Given a collection of subsets $S_1, S_2, \ldots, S_m$ of a set S, what is the size of the smallest subset of S that intersects every subset S_i?

3Color: Given an undirected graph G, can its vertices be colored with three colors, so that every edge touches vertices with two different colors?

ChromaticNumber: Given an undirected graph G, what is the minimum number of colors needed to color its vertices, so that every edge touches vertices with two different colors?

MaxCut: Given a graph G, what is the size (number of edges) of the largest bipartite subgraph of G?

HamiltonianCycle: Given a graph G, is there a cycle in G that visits every vertex exactly once?

HamiltonianPath: Given a graph G, is there a path in G that visits every vertex exactly once?

TravelingSalesman: Given a graph G with weighted edges, what is the minimum total weight of any Hamiltonian path/cycle in G?

SubsetSum: Given a set X of positive integers and an integer k, does X have a subset whose elements sum to k?

Partition: Given a set X of positive integers, can X be partitioned into two subsets with the same sum?

3Partition: Given a set X of n positive integers, can X be partitioned into $n/3$ three-element subsets, all with the same sum?

Minesweeper: Given a Minesweeper configuration and a particular square x, is it safe to click on x?

Tetris: Given a sequence of N Tetris pieces and a partially filled $n \times k$ board, is it possible to play every piece in the sequence without overflowing the board?

Sudoku: Given an $n \times n$ Sudoku puzzle, does it have a solution?

KenKen: Given an $n \times n$ Ken-Ken puzzle, does it have a solution?